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Question: How to construct the real number system? $=: \mathbb{R}$

① Algebraic axioms.

recall: $A_1 - A_4$ & $M_1 - M_4$ & D.

Addition Multiplication Distribution

② Ordering axioms.

There exists a nonempty set $\mathbb{P} \subset \mathbb{R}$ s.t.

① $\forall a \in \mathbb{R}$, then either $a=0$, $a \in \mathbb{P}$ or $-a \in \mathbb{P}$.

② if $a, b \in \mathbb{P}$, then $a+b \in \mathbb{P}$

③ if $a, b \in \mathbb{P}$, then $a \cdot b \in \mathbb{P}$.

Then define $a > b$ as $a - b \in \mathbb{P}$.

③ Completeness axioms.

$\boxed{\text{For every non-empty subset } S \text{ which is bounded from above, } \sup S \text{ exists.}}$

Definition: Let X be a non-empty subset of $\mathbb{R}$.

- $u \in \mathbb{R}$ is called an upper bound of X if $x \leq u$ for any $x \in X$.
- X is called bounded from above if it has an upper bound.
- $\sup X$ is defined to be the least upper bound of X, i.e.

$$\begin{cases} \sup X \geq x \text{ for all } x \in X \ \& \\ \sup X \leq u \text{ whenever } u \text{ is an upper bound of } X. \end{cases}$$

Hence, two steps are needed to prove $\alpha = \sup X$.
First, one has to show α is an upper bound.
Next, one needs to show α is the least among all possible upper bounds.

[lower bound, infimum are defined similarly]

Exercise:

① Let $X = [0, 1)$. Prove $\sup X = 1$.

Need to check:

① 1 is an upper bound of X

② 1 is the least upper bound.

Proof: ① is trivial. Suffices to check ②.

Let s be an upper bound of X. Then $s \geq 0$.

If $s \geq 1$, then we're done.

If $0 \leq s < 1$, choose $x = s + \frac{1-s}{2} = \frac{1+s}{2} < 1$.

Hence $x \in X$ but $x > s$

Contradiction to the fact that s is an upper bound. $\square$

② $X = \{-\frac{1}{n} : n \in \mathbb{N}\}$. Prove $\sup X = 0$.

Recall: Archimedean Property:

If $x \in \mathbb{R}$, then $\exists\, n \in \mathbb{N}$ s.t. $x \leq n$.

Proof: ① Check 0 is an upper bound of X. (Trivial)

② Suppose s be an upper bound of X.

If $s \geq 0$, then we're done.

If $s < 0$, we want to find some $x \in X$ s.t. $x > s$.

[then contradiction arises]

By Archimedean Property, $\exists\, n \in \mathbb{N}$ s.t. $-\frac{1}{s} \leq n$.

Then $s \leq -\frac{1}{n} < -\frac{1}{n+1}$ since $s<0$. But $-\frac{1}{n+1} \in X$.

$\square$

③ Prove $\sup(A+B) = \sup(A) + \sup(B)$, $A, B \subset \mathbb{R}$,

and $A+B := \{a+b : a \in A, b \in B\}$.

Need to prove: ① $\sup(A+B) \le \sup(A) + \sup(B)$ &

② $\sup(A+B) \ge \sup(A) + \sup(B)$.

Proof: ① $\forall a \in A, b \in B$, we have $a \le \sup(A)$ and $b \le \sup(B)$.

Hence $a+b \le \sup(A) + \sup(B)$.

Hence $\sup(A) + \sup(B)$ is an upper bound of $A+B$.

Hence $\sup(A+B) \le \sup(A) + \sup(B)$.

② Fix $a \in A$. Then $\forall b \in B$,

$a+b \le \sup(A+B) \Rightarrow b \le \sup(A+B) - a$.

Hence $\sup(A+B) - a$ is an upper bound of B.

Then $\sup(B) \le \sup(A+B) - a$.

Therefore, $a \le \sup(A+B) - \sup(B)$. is true $\forall a \in A$.

Hence $\sup(A+B) - \sup(B)$ is an upper bound of A.

Thus $\sup(A) \le \sup(A+B) - \sup(B)$.

$\square$